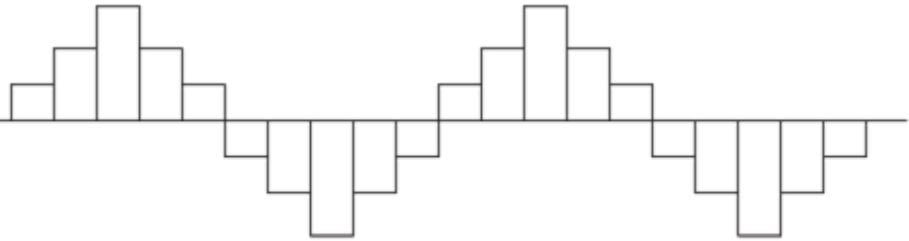
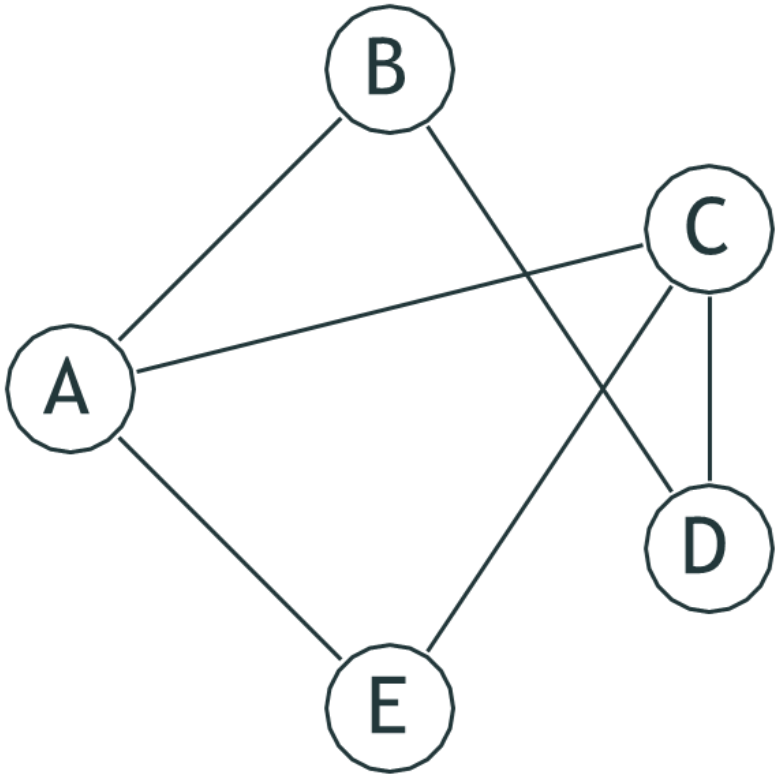
CSE1101 Discrete Mathematics for Computer Science



Graph Theory

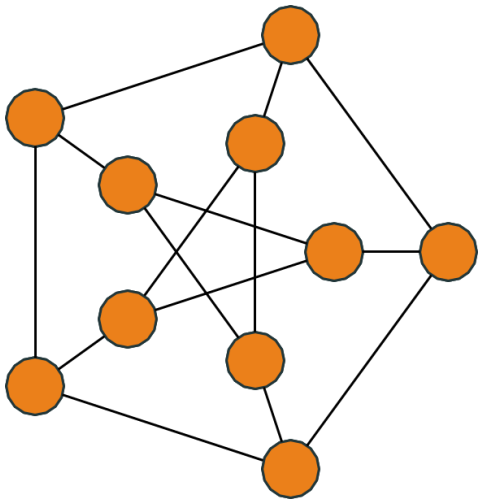
Warm Up: Airlines Graph

- Consider a small country with five cities: A, B, C, D, E.
- There are six flights:
- A – B, A – C, A – E, B – D, C – D, C – E.
- Is there a direct flight from A to D? With one stop? With two stops?

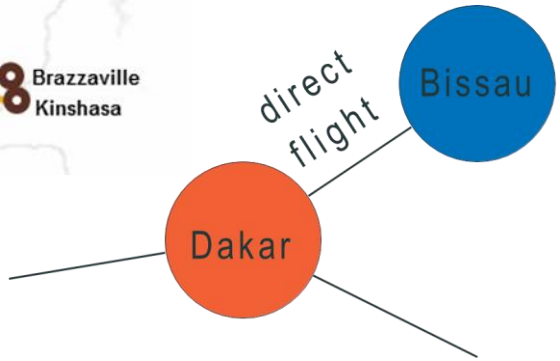


Graph Definition

- A graph represents a set of objects and relations between pairs of objects
- A Graph $G = (V, E)$ where V represents vertices (objects) and E represents edges (relationships)

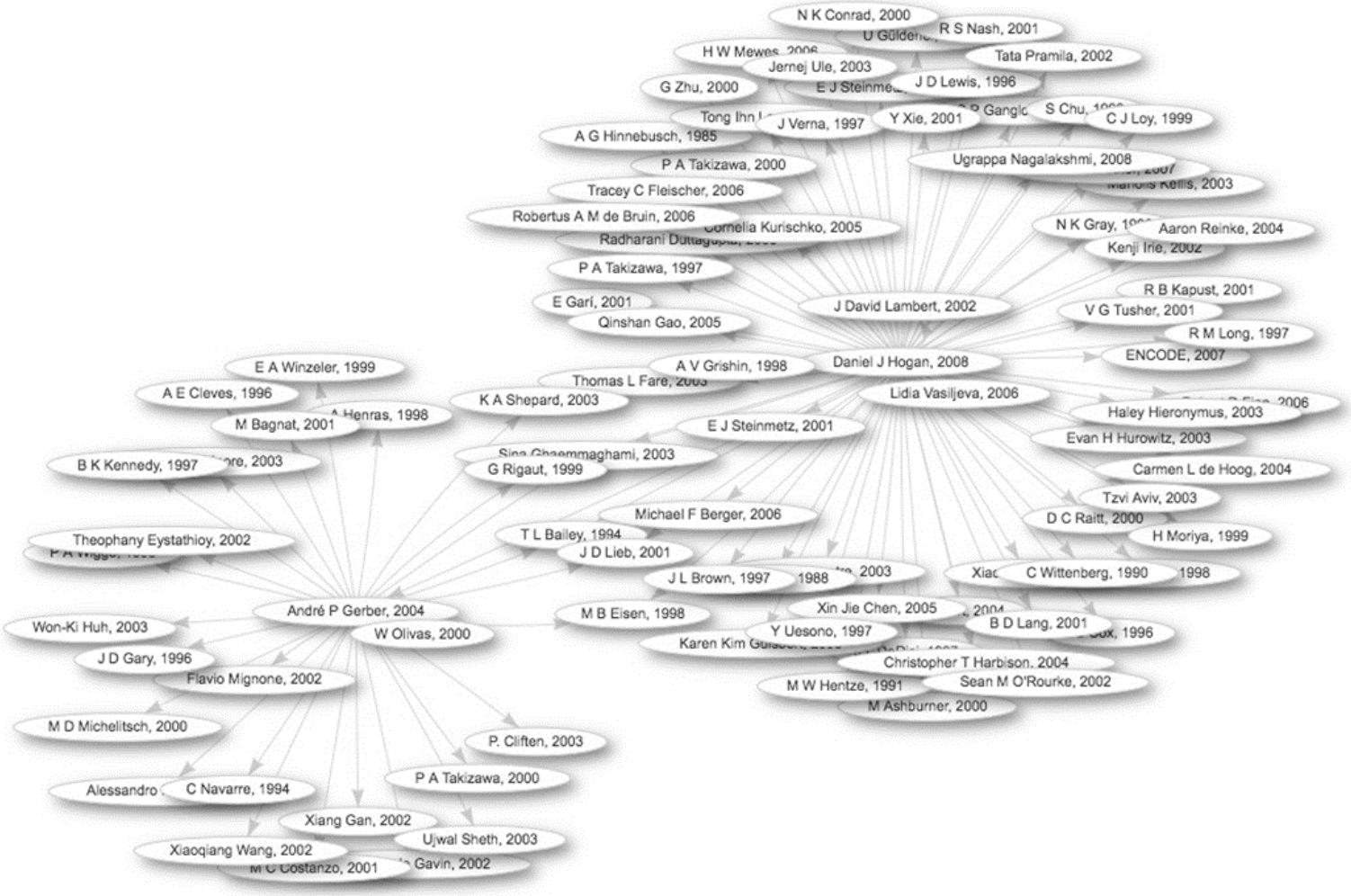


Example: Airline Graph



Example: Facebook Graph

Example: Citations of a Paper



Vocabulary

- We can name individual vertices and edges
- **Connects** u and v
- u and v are **End Points** of e
- u and e are **Incident**
- u and v are **Adjacent**
- u and v are **Neighbors**

Drawing a Graph

- Objects: $\{A, B, C, D\}$
- Relations: $\{(A, C), (D, A), (B, D), (C, B)\}$

Drawing a Graph

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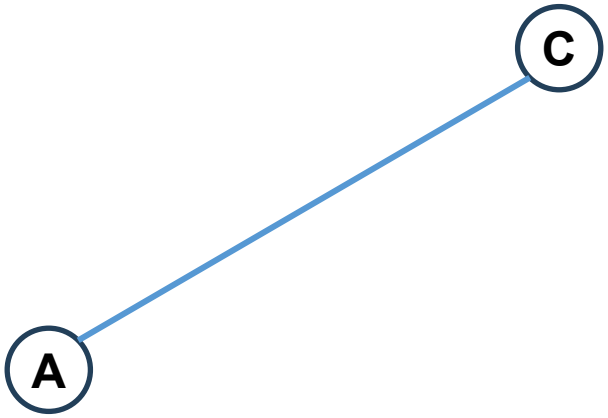
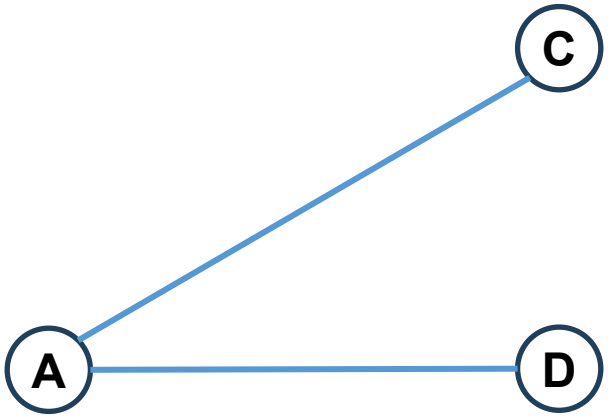
- Relations: $\{(\boldsymbol{A, C}), (D, A), (B, D), (C, B)\}$



Drawing a Graph

- Objects: $\{A, B, C, D\}$
- Relations: $\{(A, C), (\boldsymbol{D, A}), (B, D), (C, B)\}$



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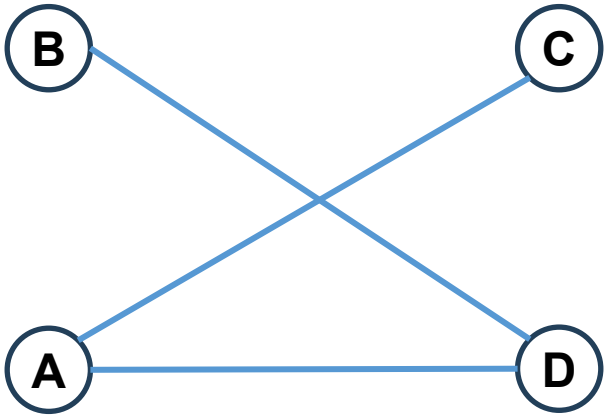

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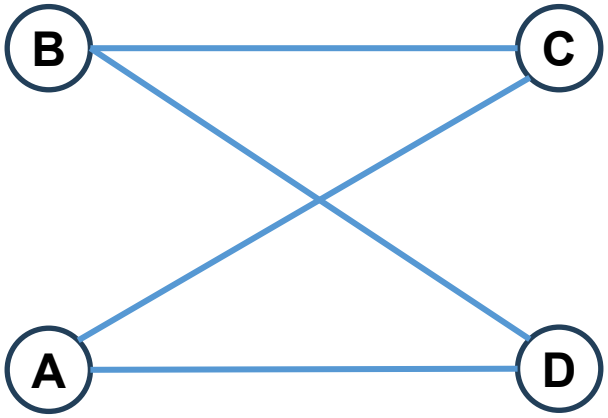

Drawing a Graph

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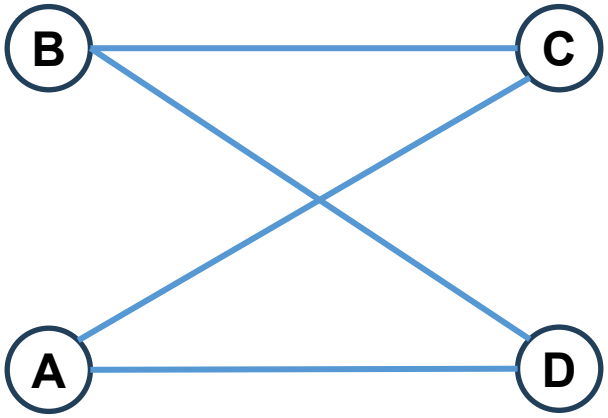
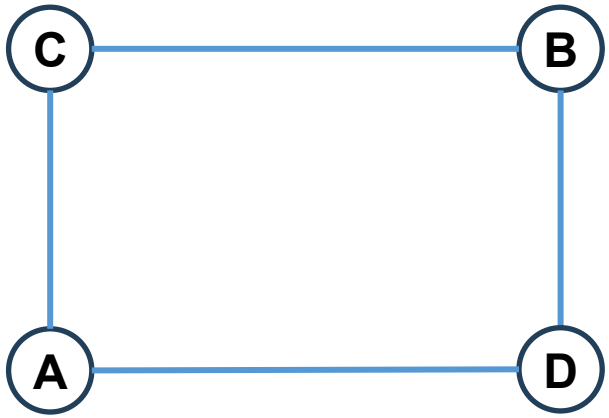
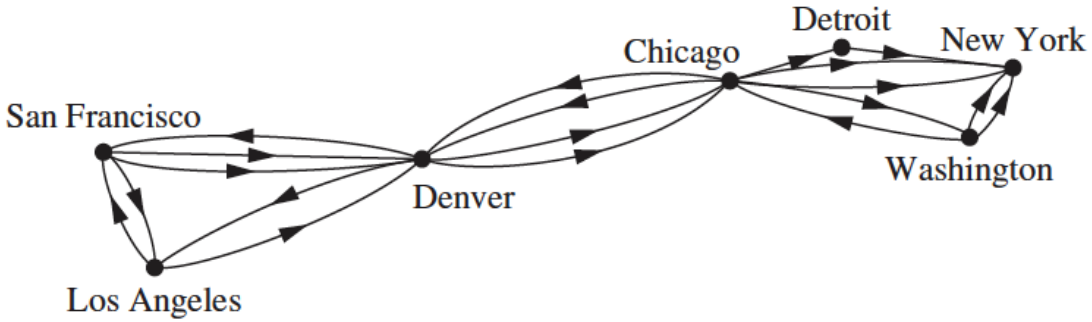




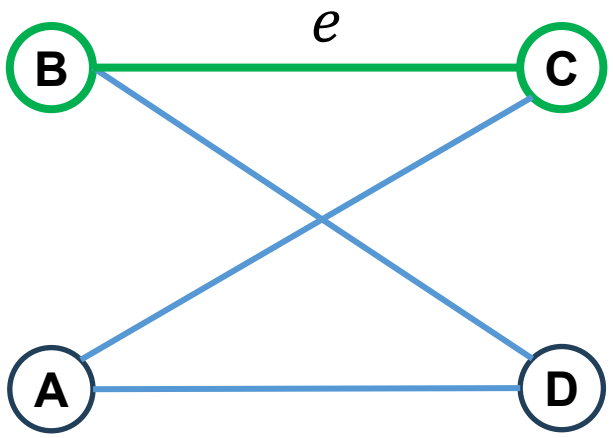
Directed Graph

- It is often convenient to consider **Directed Edges (Arcs)**
- There is a flight from A to B, but not the other way around
- Such a graph is called a Directed Graph or digraph for short.



Basic Graph Terminology

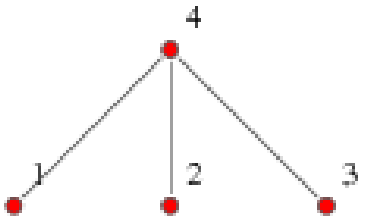
- Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G.

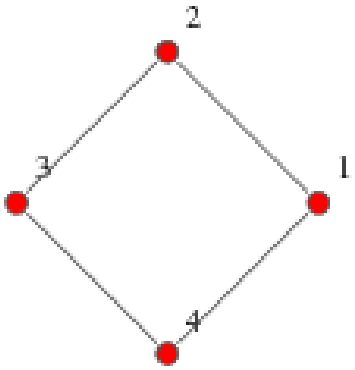


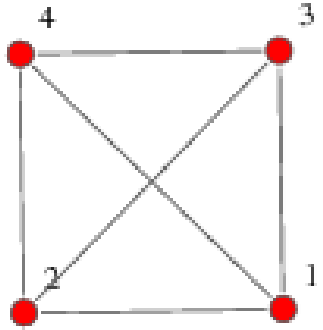
Vertex B is adjacent to C, however, B is not adjacent to A.

Drawing a Graph

- An adjacency matrix can represent a graph



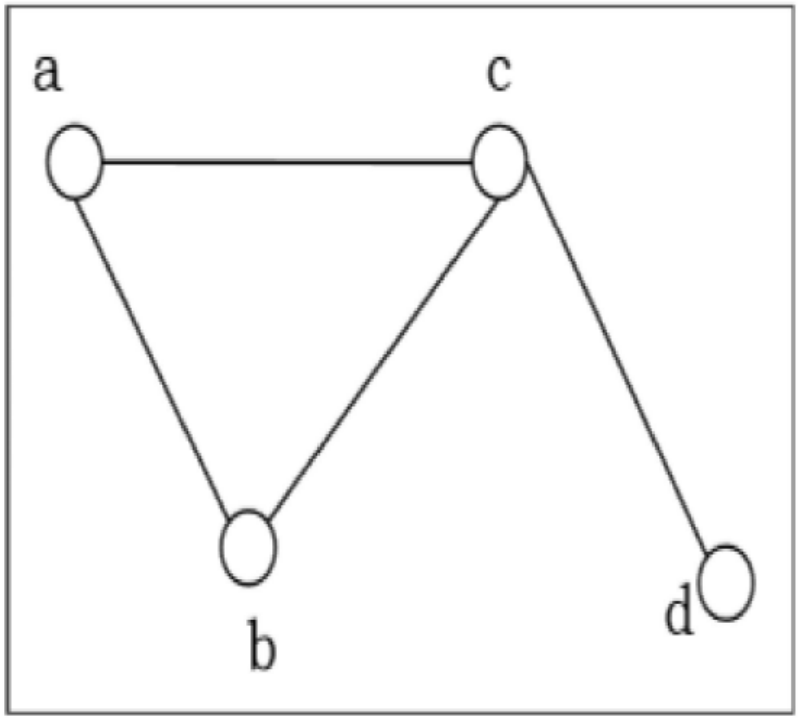




$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Drawing a Graph

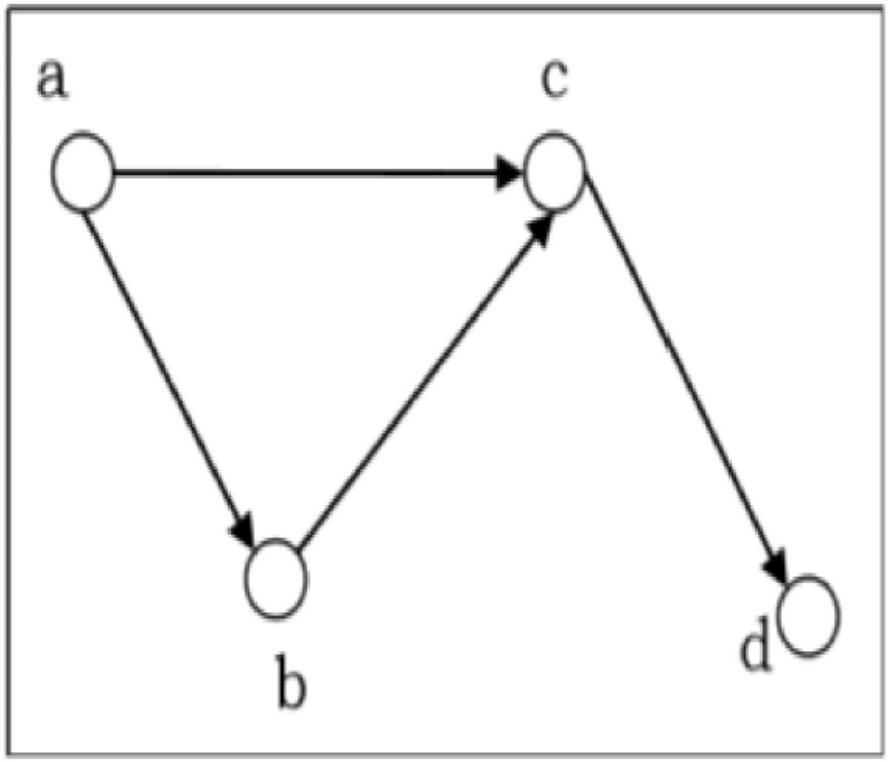
- Adjacency Matrix of an Undirected Graph



	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Drawing a Graph

- Adjacency Matrix of a Directed Graph

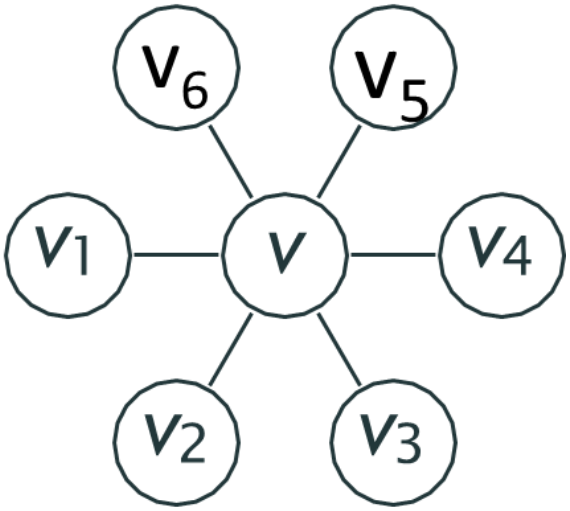


	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	0	0	0	1
d	0	0	0	0

Degree of a Vertex

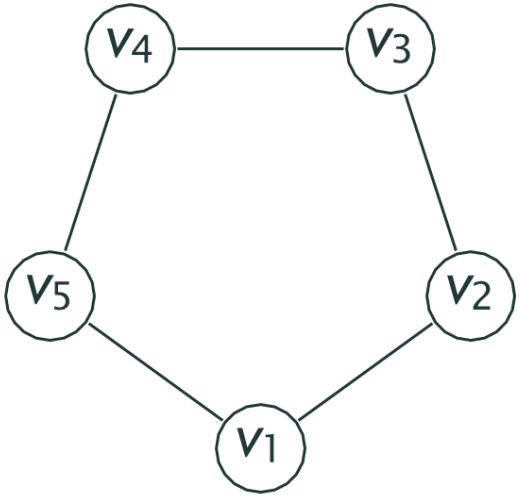
- The Degree of a vertex is the number of its incident edges
- I.e., the Degree of a vertex is the number of its neighbors
- The degree of a vertex v is denoted by deg(v)
- The degree of a graph is the maximum degree of its vertices

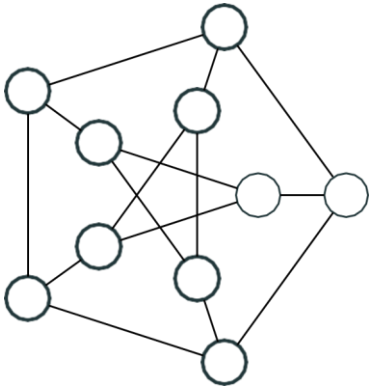
Degree of a Vertex



- The degree of v is 6: deg(v) = 6
- The degree of v_6 is 1: deg(v_6) = 1
- The degree of the graph is 6

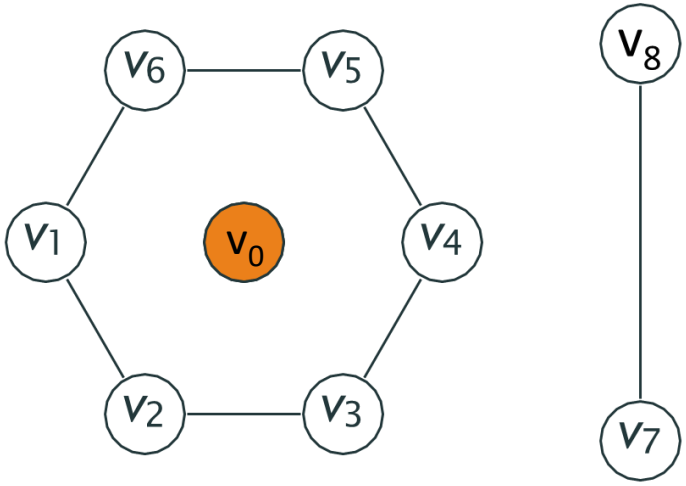
Degree of a Vertex – *k*-regular graphs





- The degree of every vertex is 2: $\forall i \deg(v_i) = 2$
- If **each vertex** of the graph has the **same degree** k the graph is called a k-regular graph
- The graph itself is said to have **degree** k

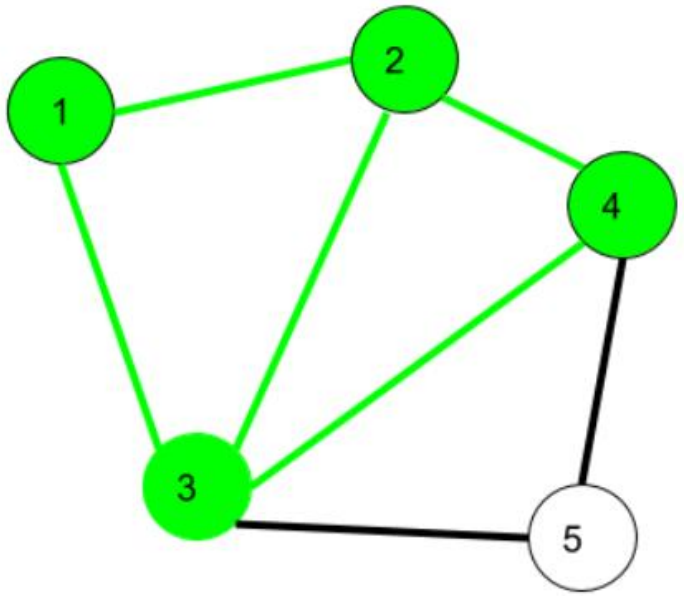
Degree of a Vertex – isolated vertex



- $\deg(v_2) = 2$
- $\deg(v_8) = 1$
- $\deg(v_0) = 0$
- v_0 is an Isolated Vertex

Walks

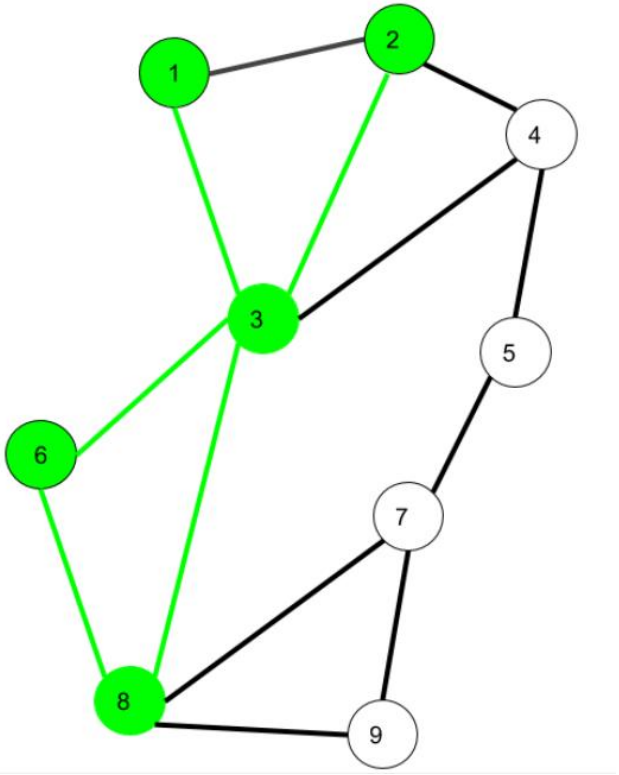
- A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk
- Vertices can be repeated
- Edges can be repeated



1->2->3->4->2->1->3 is a walk

Trail

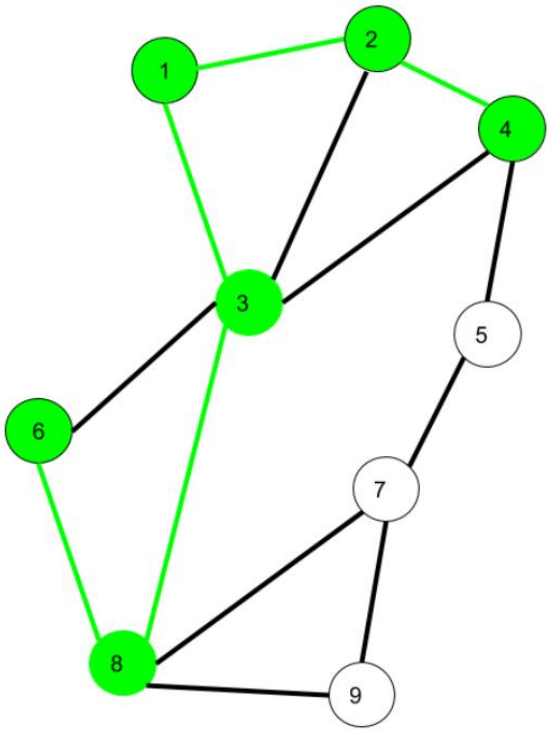
- A traversal in which no edge is repeated is a trail
- Vertices can be repeated
- Edges not repeated



1->3->8->6->3->2 is trail

Path

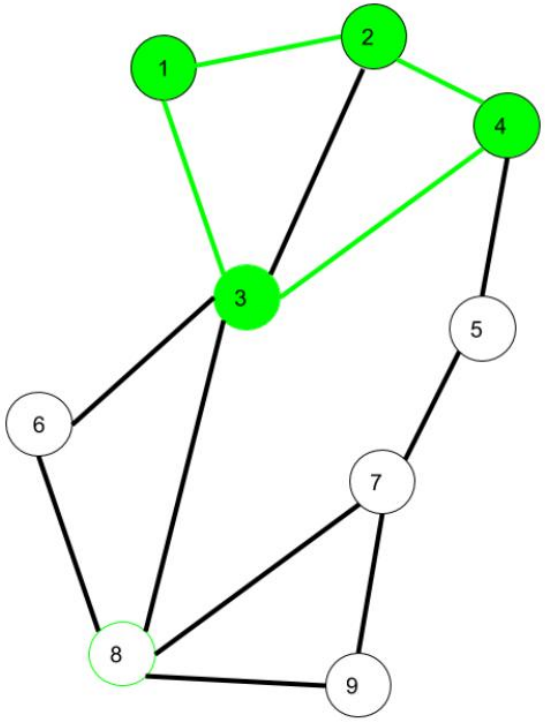
- A traversal in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge.
- Vertex not repeated
- Edge not repeated



6->8->3->1->2->4 is a Path

Cycle

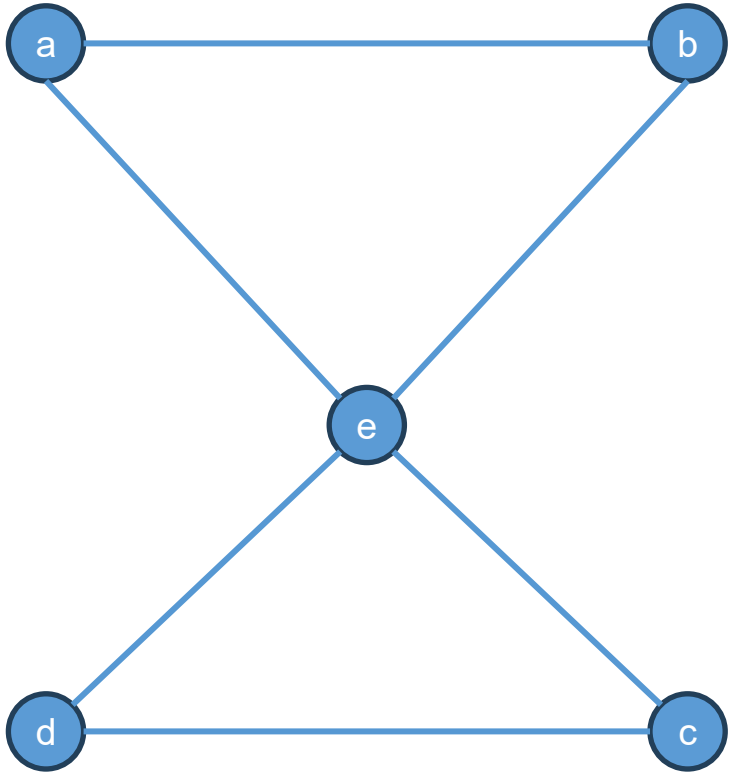
- Traversing a graph such that we do not repeat a vertex nor we repeat an edge but the starting and ending vertex must be same
- Vertices not repeated
  - - except for starting and ending
- Edges not repeated

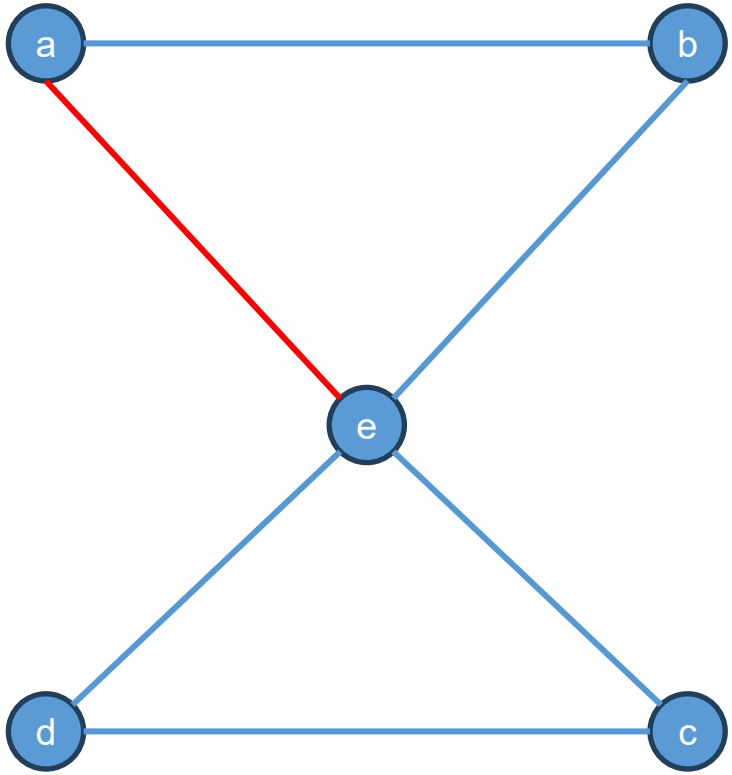


1->2->4->3->1 is a cycle

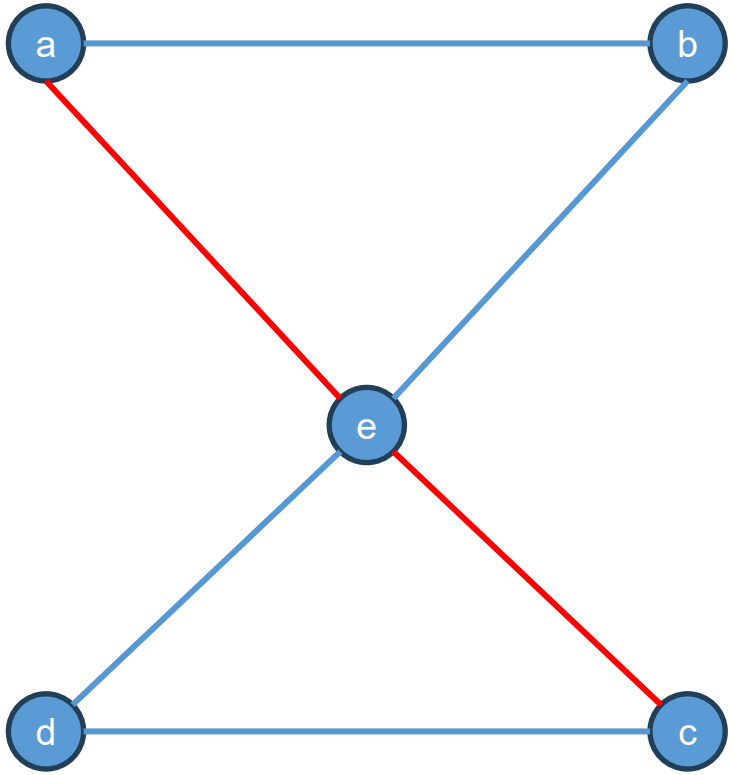
Euler Paths and Circuits

- An Euler Path visits every **edge** once
- An Euler Circuit (or cycle) visits every **edge** AND begins and ends at the same **vertex**
- An Euler Circuit is still an Euler Path – it is a special type of Euler Path
- Rules:
  - An Euler Circuit is possible if every vertex has an even degree
  - No Euler Path is possible if more than two vertices have an odd degree

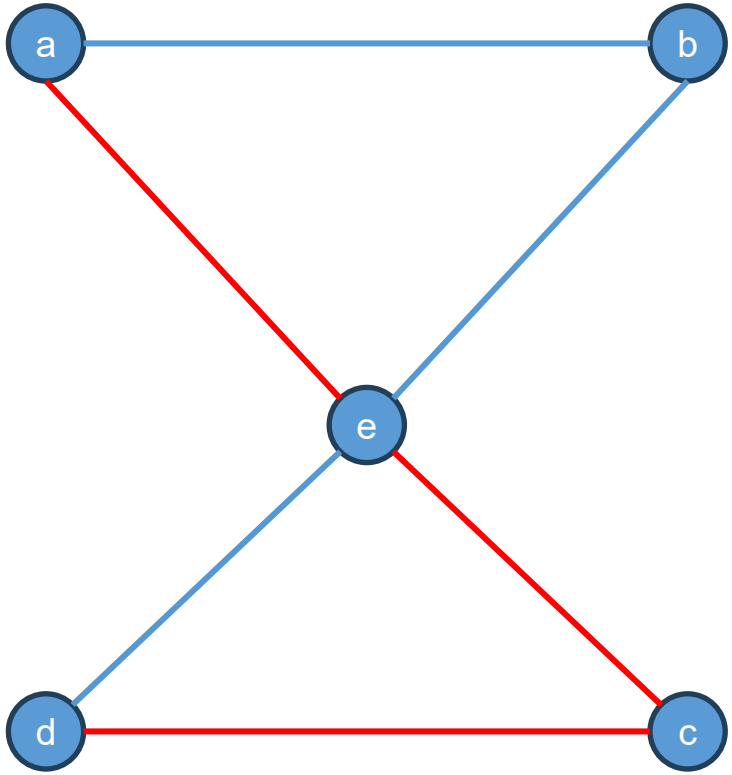

a
b
e
d
c



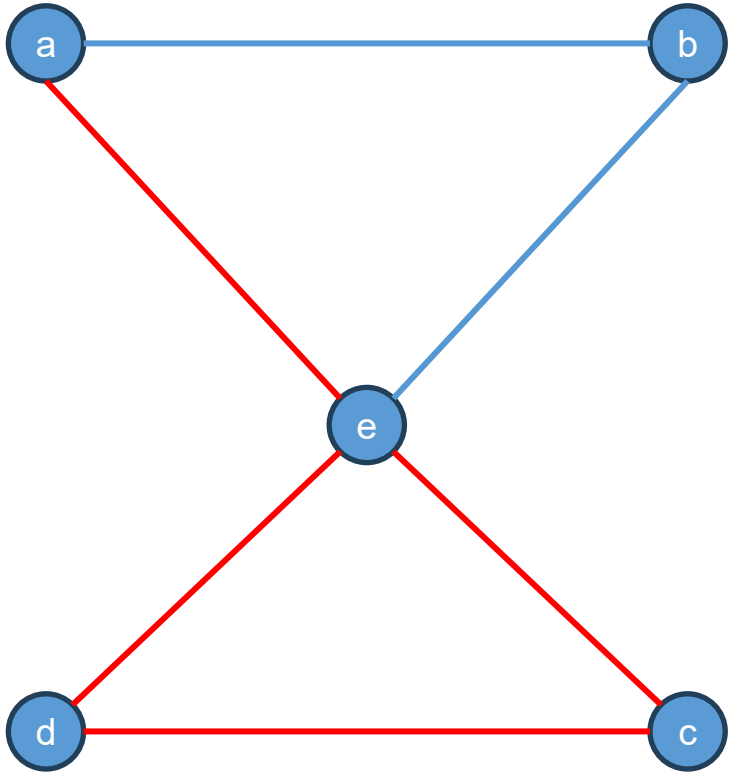
a->e



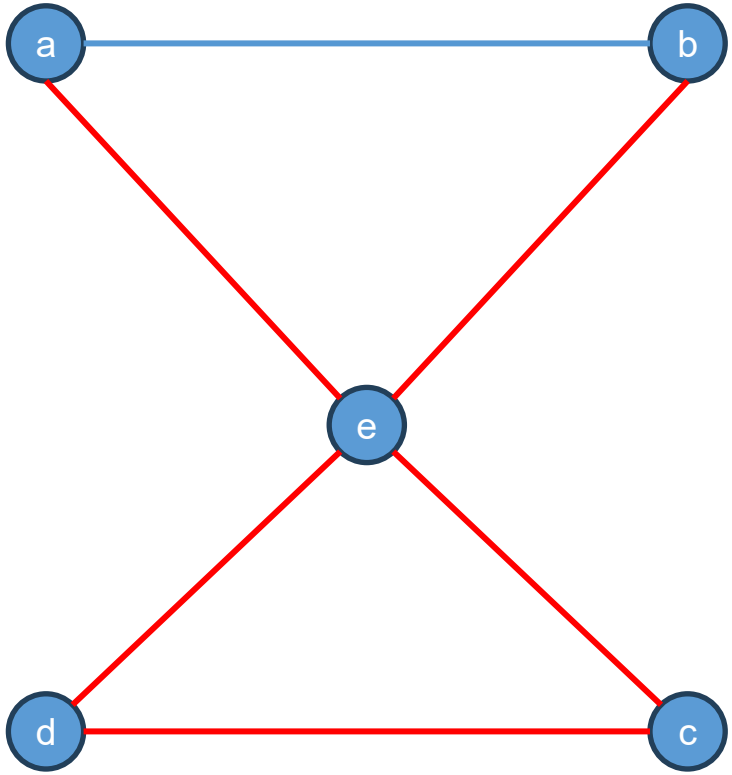
a->e->c



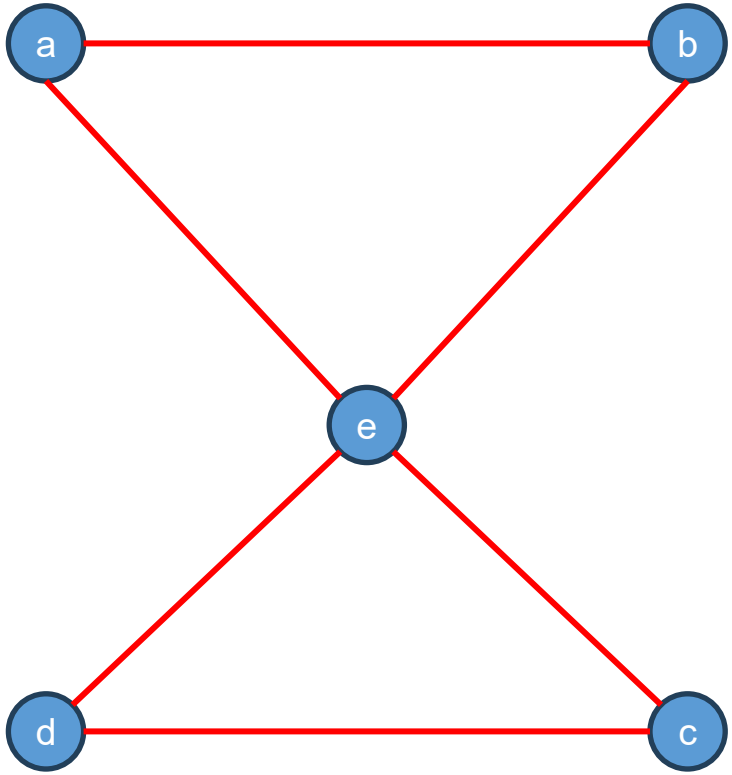
a->e->c->d



a->e->c->d->e

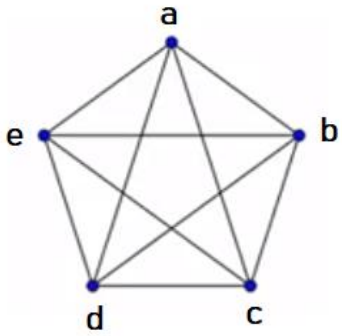


a->e->c->d->e->b



a->e->c->d->e->b->a

Euler Paths and Circuits

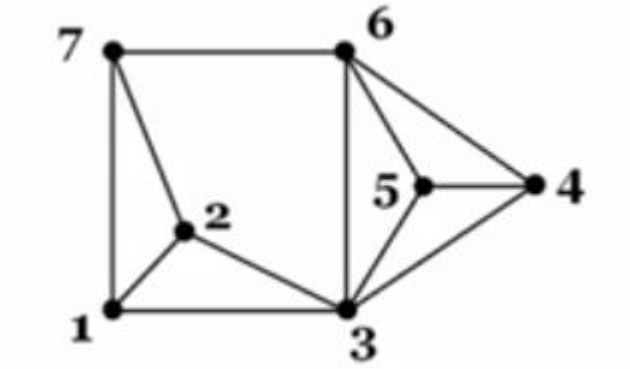


Euler Path/Circuit:
b -> e -> c -> a -> d -> b ->a -> e -> d -> c -> b

- No Euler Path is possible if more than two vertices have an odd degree
- 4-regular graph (vertices of degree 4)
- Therefore, Euler Path is possible

- An Euler Circuit is possible if every vertex has an even degree
- 4-regular graph (vertices of degree 4)
- Therefore, Euler Circuit is possible

Euler Paths and Circuits



deg(1)	3
deg(2)	3
deg(3)	5
deg(4)	3
deg(5)	3
deg(6)	4
deg(7)	3

No Euler Path is possible <u>if more than two vertices have an odd degree</u>

An Euler Circuit is possible <u>if every vertex has an even degree</u>

6 vertices have an odd degree

No Euler path is possible. No Euler circuit is possible